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Space average and wave interference in vibration conductivity

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Abstract

Plane wave interference and space averages play a significant role in the derivation of some vibration conductivity equations that are becoming more and more popular in modelling vibroacoustic problems. Particularly, the thermal approach and the modified vibration conductivity equations are here considered with the aim of establishing similarities and/or differences between them and stating relevant consequences. It is shown by formal developments that the thermal equation is obtained under the assumption of performing appropriate space averages, while the modified vibration conductivity equation does not need, in some cases, this condition. It is discussed, however, that in practical applications the conditions of validity of both approaches are quite similar.

1. Introduction

Nowadays, the interest toward high-frequency problems is becoming more and more important and demanding.

In the range of high frequencies, statistical energy analysis (SEA) is the recognized mother, although it does not yet meet a general agreement for several reasons that are recognized worldwide. In the last decade, a few alternative techniques have been introduced with the aim of overcoming some limitations of SEA while providing solutions of higher informative content. Among them are the following:

- the thermal analogy or heat conductivity or vibration conductivity method;
- the wave intensity analysis (WIA);

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- the envelope approach;
- the modified heat conductivity method.

Unlike SEA, all these methods share the possibility of providing a local response of the system with a low input requirement while keeping, as in SEA, the advantage of a low computational effort. The development of these techniques has reached different levels. WIA, which is in fact a generalization of SEA, is a basic work and actually provides theoretical elements that are used in other energetic techniques: recently it was applied successfully to predict the vibration levels in assemblies of plate panels [1]. The envelope method that, unlike most of the other methods, is not an energetic approach, provides a complex envelope displacement solution of the high-frequency problem from which the physical solution can be recovered: however, so far this method has been reliably applied to one-dimensional systems only, while the extension to more complex systems is yet under development [2].

The thermal analogy is an approach that has met a high interest in the scientific community, in that it is the one originally developed several years ago as an extension of the energy balance SEA laws, with the advantage, with respect to SEA, of providing a "local" response along the system. Actually, in recent years the thermal analogy showed some limitations and met some criticisms, especially when applied to two-dimensional systems.

Recently some authors, by the use of a wave approach and the introduction of some simplified assumptions, obtained the same heat thermal equation and another equation, here called for reference the modified heat conductivity equation, that seems to provide more reliable results, especially for two-dimensional structures. These approaches are mainly based on appropriate space averages and the assumption of neglecting the interference among propagating waves. Therefore, interference and space-averaging are critical subjects for these methods based on the energy propagation: an insight into the contribution of the wave interaction to the energy propagation mechanism is thus necessary to understand the limits of validity of the previous methods and establish the main differences between them.

2. Background on vibration conductivity

In the mentioned methods, the thermal analogy and the modified heat conductivity method, two main operations are used that have a relevant role on the whole analysis: the space average and the assumption of neglecting wave interference. Their effects are only partially evaluated: they are basic for the development of the methods, but a critical analysis is rather missing, although a serious contribution related to these points is provided by Langley in Ref. [3].

In this section these methods are revisited under this point of view.

When speaking about the thermal analogy, we refer here explicitly to the approach proposed by Ichchou et al. [4,5]. In fact, the thermal method has different derivations: it begun with a differential formulation of the SEA laws proposed by a group of Soviet scientists [6,7], became generally known with a paper by Nefske and Sung (PFFM) [8], provided interesting contributions in different works by Bernhard et al. (e.g., Refs. [9–11]), and, finally, found a more precise formulation based on the wave approach in some works by Jezequel et al. (e.g., Refs. [4,5,12]). The concept of wave interference, often considered in SEA and other energetic approaches, is

introduced in the wave approach proposed by these last authors, beyond the assumptions used to derive the heat equation:

- linear, elastic systems;
- small hysteretic damping;
- steady state condition under harmonic excitation;
- near fields are neglected.

The new assumption used by Jezequel et al. claims that the interference among propagating waves must be neglected, i.e., no correlation exists between plane waves. Under the above assumptions, the time and space average energy parameters such as the active energy flow and energy density can be obtained from a linear superposition of the partial energies associated to each wave direction.

In Ref. [3] Langley presents a deep discussion on the validity of the thermal approach, showing that the heat conductivity equation can be derived under the assumption of neglecting the correlation among plane waves. However, he points out that the thermal analogy equation leads, for cases of cylindrical symmetry, to an energy density decreasing as $1/\sqrt{r}$, versus a correct energy solution that is represented by a dependency on $1/r$, given by the square of the Bessel function of zeroth order J_0. This statement is also recalled by Le Bot in Refs. [13,14] when presenting his integral model of modified heat conductivity. Therefore, it is concluded that the thermal analogy is strictly not correct, although, under particular conditions, it can provide acceptable results.

In Ref. [15] Carcaterra and Sestieri have derived exact time-average energy equations for the free problem of one- and two-dimensional systems, while in Ref. [16] Carcaterra and Adamo have discussed, both theoretically and experimentally, the role of wave interference in beams and plates, using both time and space averages.

In the present paper the aim is to show: (i) the derivation of a general exact energy equation, valid for systems governed by the Helmholtz equation; (ii) which of the terms of the exact energy equations account for the interference effects; (iii) whether and how, by starting from the exact energy equation, first presented in Ref. [15] for one-dimensional structures and here extended to two-dimensional systems, the thermal equation can be determined by introducing appropriate space average hypotheses and/or by neglecting the interference of plane waves; (iv) whether the condition of space average and/or uncorrelated waves is a necessary condition to yield the modified heat equation or it can be determined by less strict assumptions. Finally, some comments about the implications of interference effects will be provided.

2.1. Reference balance equations

2.1.1. Exact energy equation for second order systems governed by the Helmholtz equation

The energy equation determined in Ref. [15], valid for one-dimensional systems characterized by second order equations or for the farfield components of flexural structures, can be extended to two- and three-dimensional systems. To this aim, systems governed by the Helmholtz equation (membranes, farfield components of flexural plates, acoustic fields) are considered in which a

harmonic vibration $\mathbf{w}(\mathbf{x})e^{j\omega_0 t}$ (wavenumber k_0) is excited, i.e.,

$$\nabla^2 w(\mathbf{x}) + k_0^2 w(\mathbf{x}) = 0, \tag{1}$$

w is a real vector displacement in the absence of damping.

Multiplying Eq. (1) by w, gives

$$w \cdot \nabla^2 w + k_0^2 w \cdot w = 0. \tag{2}$$

Considering the time-average kinetic energy density, one can generally write

$$\langle T \rangle = \kappa_T w \cdot w = \kappa_T |w|^2, \tag{3}$$

where κ_T is a constant dependent on the system considered: e.g., for a membrane or a plate $\kappa_T = m'\omega_0^2/4$, m' being the mass per unit area. $\langle\ \rangle$ denotes the time average and $k_0 = \omega_0/c$, c being the phase velocity of the system considered. Since:

$$grad \langle T \rangle = 2\kappa_T \, grad \, w \cdot w \tag{4}$$

it is obvious that

$$\nabla^2 \langle T \rangle = 2\kappa_T w \cdot \nabla^2 w + 2\kappa_T |grad \, w|^2.$$

Thus,

$$w \cdot \nabla^2 w = \frac{1}{2\kappa_T} \nabla^2 \langle T \rangle - |grad \, w|^2. \tag{5}$$

By substituting Eqs. (3) and (5) into Eq. (2), one obtains

$$\nabla^2 \langle T \rangle - 2\kappa_T |\, grad \, w|^2 + 2k_0^2 \langle T \rangle = 0 \tag{6}$$

and expressing $|grad \, w|$ in terms of $\langle T \rangle$, using Eq. (4), i.e.,

$$|grad \, w|^2 = \frac{|grad \langle T \rangle|^2}{4\kappa_T \langle T \rangle}$$

one finally obtains the exact equation for the kinetic energy density, i.e.,

$$\nabla^2 \langle T \rangle - \frac{|grad \langle T \rangle|^2}{2\langle T \rangle} + 2k_0^2 \langle T \rangle = 0. \tag{7}$$

Since in the literature the energy balance equations are often written in cylindrical co-ordinates (r, θ) (two-dimensional case), from Eq. (7) one has, in this case

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \langle T \rangle}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \langle T \rangle}{\partial \theta^2} - \frac{1}{2\langle T \rangle}\left\{\left(\frac{\partial \langle T \rangle}{\partial r}\right)^2 + \frac{1}{r}\left(\frac{\partial \langle T \rangle}{\partial \theta}\right)^2\right\} + 2k_0^2 \langle T \rangle = 0. \tag{8}$$

If θ is eliminated, i.e., a case of cylindrical symmetry is studied, Eq. (8) becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \langle T \rangle}{\partial r}\right) - \frac{1}{2\langle T \rangle}\left(\frac{\partial \langle T \rangle}{\partial r}\right)^2 + 2k_0^2 \langle T \rangle = 0. \tag{9}$$

It can be shown that, for a membrane where the vector $\mathbf{w}$ becomes simply a scalar w, Eq. (7) can be expressed in terms of the kinetic and potential energy densities, obtaining a simpler form. In

fact, for the membrane, the potential energy density is

$$\langle U \rangle = \tfrac{1}{4} t \, |grad \, w|^2,$$

where t is the traction per unit length. Since $c = \sqrt{t/m'}$ is the wave speed in the membrane, one can easily determine the relationship

$$\langle U \rangle = \frac{|grad \langle T \rangle|^2}{4k_0^2 \langle T \rangle},$$

so that one can write, for the membrane

$$\Rightarrow \nabla^2 \langle T \rangle + 2k_0^2(\langle T \rangle - \langle U \rangle) = 0. \tag{10}$$

(For flexural systems, Eq. (10) cannot be directly written in terms of potential energy only, but rather a more complex expression is obtained [15].)

2.1.2. Established energy equations used in the literature

The vibration conductivity equation (and the modified heat equation) is usually written in terms of the time and "space-averaged" total energy density $\bar{E}$ as follows:

$$\nabla^2 \langle \bar{E} \rangle = 0 \quad \text{for undamped systems} \tag{11}$$

or

$$\nabla^2 \langle \bar{E} \rangle - \left(\frac{\eta\omega}{c_g}\right)^2 \langle \bar{E} \rangle = 0 \quad \text{for damped systems.} \tag{12}$$

A discussion on the meaning and kind of space average will be presented in the following sections.

In cylindrical co-ordinates, Eq. (11) becomes

$$\frac{1}{r}\frac{\partial \langle \bar{E} \rangle}{\partial r} + \frac{\partial^2 \langle \bar{E} \rangle}{\partial r^2} = 0 \quad \text{or} \quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \langle \bar{E} \rangle}{\partial r}\right) = 0. \tag{13}$$

Another equation that is considered in the literature and used successfully in a recent integral formulation presented by Le Bot [14] is the undamped modified heat conductivity equation determined in Refs. [4,5]. This is written in cylindrical co-ordinates as a first order differential equation and is valid, in principle, for cylindrical symmetry. By using simple relationships, it can be transformed into a second order differential equation that reads

$$\frac{2}{r}\frac{\partial \langle \tilde{E} \rangle}{\partial r} + \frac{\partial^2 \langle \tilde{E} \rangle}{\partial r^2} = 0 \quad \text{or} \quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \langle \tilde{E} \rangle}{\partial r}\right) + \frac{1}{r}\frac{\partial \langle \tilde{E} \rangle}{\partial r} = 0. \tag{14}$$

This equation is obviously non-thermal, as can be seen by comparing it with Eq. (13). The tilde implies some kind of operation that is not stated explicitly here, but considered in the next sections.

3. Spherical mean of the energy equation

It will be shown in Section 3.2 how an exact general time and space-averaged energy equation can be determined starting from Eq. (7). This is obtained by the concept of spherical mean, that is a way to perform space moving averages of physical quantities: it permits one to perform simple calculations on the space derivatives of an average function [17]. Moreover, the spherical mean introduces the size (radius) ρ of the domain over which the average is performed that, according to Ref. [16], plays a relevant role in attenuating the interference effects. This is specifically shown in Section 3.2, where there is a discussion as to how the interference terms tend to vanish as the size of the integration domain increases. It is worthwhile to point out that the results of the spherical mean could be assimilated to those obtained by any other type of space average, with ρ equivalent to the characteristic dimension of the space-average domain.

For two-dimensional problems, the moving spherical mean over the interior of a circle of radius ρ is given by

$$\hat{f}(\mathbf{x},\rho)=\frac{1}{\pi\rho^2}\int_0^{\rho}\int_0^{2\pi} f[\mathbf{x}+\mathbf{n}(\theta)\mu]\mu\,\mathrm{d}\mu\,\mathrm{d}\theta, \tag{15}$$

while the spherical mean over the boundary of the circle is

$$\hat{f}_{\partial}(\mathbf{x},\rho)=\frac{1}{2\pi}\int_0^{2\pi} f[\mathbf{x}+\mathbf{n}(\theta)\rho]\,\mathrm{d}\theta. \tag{16}$$

For a better comprehension of the spherical average, see Fig. 1.

Using these definitions, one can simply obtain the following properties:

$$\text{(p1)}\quad \widehat{\nabla^2 f}=\nabla^2\hat{f},\quad \text{(p2)}\quad \nabla^2\hat{f}=\frac{2}{\rho}\frac{\partial\hat{f}_{\partial}}{\partial\rho}.$$

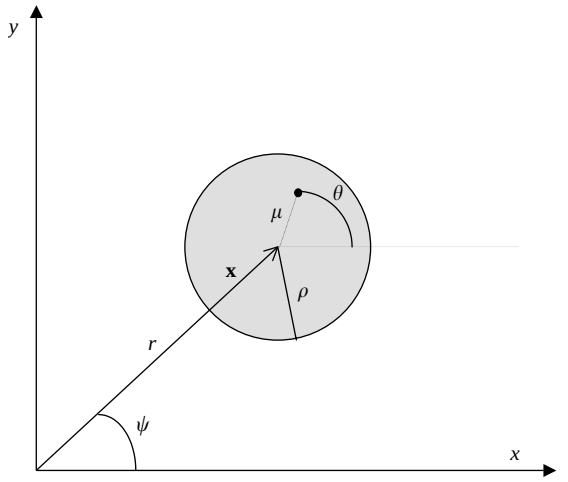

Fig. 1. Spherical mean over the inner of the circle.

These relationships have important implications. When a spherical mean is performed on Eqs. (7) and property (p1) is used, one obtains

$$\nabla^2 \langle \hat{T} \rangle + \Phi_{int} = 0 \tag{17}$$

or, by using property (p2), one has

$$\frac{2}{\rho} \frac{\partial \langle \hat{T} \rangle_{\partial}}{\partial \rho} + \Phi_{int} = 0, \tag{18}$$

where Φ_{int} is an interference term (as is shown later) and given by

$$\Phi_{int} = 2k_0^2 \langle \hat{T} \rangle - \frac{1}{\pi \rho^2} \int_0^{\rho} \int_0^{2\pi} \frac{|grad \langle T \rangle|^2}{2 \langle T \rangle} \mu \, \mathrm{d}\mu \, \mathrm{d}\theta. \tag{19}$$

In the next section the meaning of the term Φ_{int} is shown and its dependency on the radius ρ of the spherical mean.

3.1. Remarks on the expansion of the energy field in terms of plane waves

To better focus on the interference problem and have a more precise understanding of the different balance equations proposed in the literature, a brief recall to the interference and coincident energies is appropriate.

Considering the Helmholtz equation for two-dimensional systems:

$$\nabla^2 w + k_0^2 w = 0, \tag{20}$$

one can stress that a plane propagating wave

$$w(\mathbf{r}) = A \mathrm{e}^{\mathrm{j}\mathbf{k}\cdot\mathbf{r}} \tag{21}$$

with $\mathbf{r}$ being the position vector, A the wave amplitude, $j = \sqrt{-1}$, $\mathbf{k} = k_0 \mathbf{n}$, and $\mathbf{n} = [\cos\theta \ \sin\theta]^{\mathrm{T}}$, is a solution of Eq. (20). For the linearity of the problem, a more general solution can be obtained by superimposing plane waves of type (21) with different θ's [17], giving rise to a real displacement w. Thus, in integral form

$$w(\mathbf{r}) = \int_0^{2\pi} A(\theta) \mathrm{e}^{\mathrm{j}k_0 \mathbf{n}(\theta)\mathbf{r}} \, \mathrm{d}\theta \quad \text{with } A(\theta + \pi) = A^*(\theta). \tag{22}$$

This represents the cylindrical standing wave solution of the homogeneous equation, but it is not the solution for forced problems. Therefore, the energy developments based on Eq. (22) hold, for forced problems, in regions of the structure where no loads are included.

The value of integral (22) for $A(\theta) = A = const$ is proportional to the Bessel function J_0, and provides, particularly, the value $2\pi A \mathrm{J}_0(k_0 r)$.

The time-average kinetic energy per unit area of the system can be obtained as follows:

$$\begin{aligned} \langle T \rangle &= \kappa_T w^2(\mathbf{r}) \\ &= \kappa_T \int_0^{2\pi} \int_0^{2\pi} A(\theta_1) A^*(\theta_2) \mathrm{e}^{\mathrm{j}k_0 \mathbf{n}(\theta_1)\mathbf{r}} \mathrm{e}^{-\mathrm{j}k_0^* \mathbf{n}(\theta_2)\mathbf{r}} \, \mathrm{d}\theta_1 \, \mathrm{d}\theta_2, \end{aligned}$$

where $\kappa_T = m'\omega^2/4$, m' being the mass per unit area.

The energy related to interferent plane waves is represented by the integral contribution for $\theta_1 \neq \theta_2$, while the coincident energy is that associated to $\theta_1 = \theta_2$.

3.2. Asymptotic analysis of the interference term Φ_{int}

The spherical mean over the boundary of the time-average kinetic energy determined in the previous section is given by

$$\langle \hat{T}(\mathbf{x},\rho)\rangle_{\partial} = \frac{1}{2\pi}\int_0^{2\pi} \langle T\rangle[\mathbf{x}+\rho\mathbf{n}(\phi)]\,\mathrm{d}\phi$$

i.e.,

$$\langle \hat{T}(\mathbf{x},\rho)\rangle_{\partial} = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} A(\theta_1)A^*(\theta_2)\mathrm{e}^{\mathrm{j}k_0\mathbf{n}(\theta_1)[\mathbf{x}+\rho\mathbf{n}(\phi)]}\mathrm{e}^{-\mathrm{j}k_0\mathbf{n}(\theta_2)[\mathbf{x}+\rho\mathbf{n}(\phi)]}\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2\,\mathrm{d}\phi$$

or

$$\begin{aligned}\langle \hat{T}(\mathbf{x},\rho)\rangle_{\partial} = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} & A(\theta_1)A^*(\theta_2)\mathrm{e}^{\mathrm{j}k_0 r\{\cos(\theta_1-\psi)-\cos(\theta_2-\psi)\}} \\ & \times \mathrm{e}^{\mathrm{j}k_0\rho\{\cos(\theta_1-\phi)-\cos(\theta_2-\phi)\}}\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2\,\mathrm{d}\phi,\end{aligned} \tag{23}$$

where $\mathbf{x} = r[\cos\psi\ \sin\psi]^{\mathrm{T}}$, $\mathbf{n}(\theta_1) = [\cos\theta_1\ \sin\theta_1]^{\mathrm{T}}$, $\mathbf{n}(\theta_2) = [\cos\theta_2\ \sin\theta_2]^{\mathrm{T}}$, $\mathbf{n}(\phi) = [\cos\phi\ \sin\phi]^{\mathrm{T}}$, k_0 is a real wavenumber in the absence of damping, the case considered in the following developments of this paper, or $k_0 = (1+\mathrm{j}\eta/4)k_0$ for a damped system. One can now integrate such expression for $k_0\rho\to\infty$, using the asymptotic expansion up to the first order (Kelvin method). The Kelvin method or stationary phase method states that the generalized Fourier integral:

$$I(x) = \int_a^b \mathrm{e}^{x\varphi(t)}g(t)\,\mathrm{d}t$$

can be approximated for large values of x by

$$I(x)\approx\sqrt{\frac{2\pi}{x|\varphi''(t_0)|}}\,g(t_0)\mathrm{e}^{\mathrm{j}x\varphi(t_0)+\mathrm{j}(\pi/4)\ \mathrm{sign}(\varphi''(t_0))},$$

where t_0 is a stationary point of $\varphi(t)$.

In the present case the phase function is given by

$$\varphi = \cos(\theta_1-\phi)-\cos(\theta_2-\phi)$$

so that the stationary points are given by

$$t_0 = \phi_0 = \frac{\theta_1+\theta_2}{2}-\frac{\pi}{2} \quad \text{or} \quad \theta_1 = \theta_2.$$

Note that the last solution cannot be considered for the evaluation of the interference energy.

In expression (23) of the kinetic energy both the coincident contribution ($\theta_1 = \theta_2$) and the interference contribution are present. First determine the interference term. By using the

asymptotic expansion for $k_0\rho \to \infty$, i.e., for $\rho/\lambda_0 \to \infty$, one finds that

$$\langle \hat{T}_{int}(\mathbf{x},\rho)\rangle_\partial = \sqrt{\frac{1}{2\pi\rho k_0}} \int_0^{2\pi}\int_0^{2\pi} \frac{A(\theta_1)A^*(\theta_2)}{\sqrt{|f^{(0)}|}} \times e^{jk_0 r\{\cos(\theta_1-\psi)-\cos(\theta_2-\psi)\}} e^{j\{k_0\rho f^{(0)}-\mathrm{sign}(f^{(0)})\pi/4\}}\, d\theta_1\, d\theta_2,$$

where $f^{(0)} = \cos(\theta_1/2 - \theta_2/2 + \pi/2) - \cos(\theta_2/2 - \theta_1/2 + \pi/2)$. The integral, and thus $\langle \hat{T}_{int}(\mathbf{x},\rho)\rangle_\partial$, presents a dependency on ρ given by $e^{jk_0\rho}/\sqrt{k_0\rho}$.

Because of the assumption of absence of damping, the coincident term is given by ($\theta_1 = \theta_2 = \theta$):

$$\langle \hat{T}_{coin}\rangle_\partial = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} |A(\theta)|^2\, d\theta\, d\phi = const.$$

It is, evidently, $\langle \hat{T}\rangle_\partial = \langle \hat{T}_{coin}\rangle_\partial + \langle \hat{T}_{int}\rangle_\partial$, and it is trivial to prove that, for an undamped system, one has $\nabla^2\langle \hat{T}_{coin}\rangle_\partial = 0$, so that Eq. (18) can be rewritten as

$$\frac{2}{\rho}\frac{\partial\langle \hat{T}_{int}\rangle_\partial}{\partial\rho} + \Phi_{int} = 0.$$

By considering the expression of $\langle \hat{T}_{int}\rangle_\partial$, one has that its derivative shows an asymptotic dependency on $k_0\rho$ given by

$$\frac{e^{jk_0\rho}}{(k_0\rho)^{1/2}}$$

so that

$$\frac{2}{\rho}\frac{\partial\langle \hat{T}_{int}\rangle_\partial}{\partial\rho} \to 0 \quad \text{as} \quad \frac{e^{jk_0\rho}}{(k_0\rho)^{3/2}}.$$

Therefore, for $\rho \to \infty$, Φ_{int} decreases as $1/\sqrt{(k_0\rho)^3}$ and not monotonically ($e^{jk_0\rho}$). This shows that, for large values of ρ, the interference terms in the energy equation, related to Φ_{int}, tend to disappear, provided that the space averages are computed on rather large regions.

In Fig. 2 the case of a membrane is considered and the interference term averaged over the membrane domain S, and normalized with respect to the total energy of the membrane, is shown

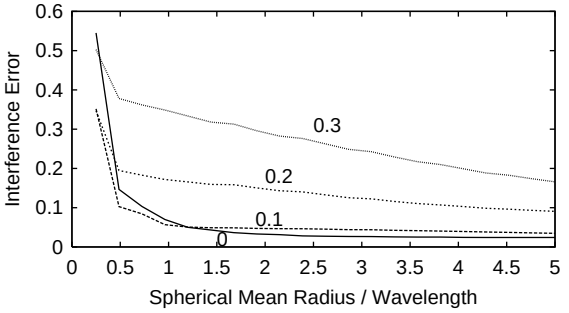

Fig. 2. Normalized interference error versus ρ/λ_0, for different values of damping η.

for different values of the ratio ρ/λ_0 and for different values of damping. The normalized error is, explicitly

$$\frac{\int_S \Phi_{int}(\mathbf{x},\rho)\,\mathrm{d}S}{\int_S \langle E\rangle(\mathbf{x})\,\mathrm{d}S},$$

where Φ_{int} is computed through Eq. (19). Although in the previous theoretical developments damping was absent, it is introduced here because the heat conductivity equation is practically meaningless if damping is zero, in that it provides a flat solution along the system. The membrane is square $1.26\ \mathrm{m} \times 1.26\ \mathrm{m}$ and excited by a harmonic point force at $x = 0.3$ m, $y = 0.1$ m generating a characteristic wavelength response $\lambda_0 = 0.14$ m, corresponding to a wavenumber $k_0 \simeq 45\ \mathrm{m}^{-1}$.

The considerations presented above on the trend of the interference terms with ρ are confirmed, while one can also observe that the interference effect is higher when increasing the damping: this would show that, when damping is larger, the direct field is dominant and, in these conditions, the vibration conductivity is more difficultly achieved.

In Fig. 3 the physical kinetic energy of the considered membrane is presented for $\eta = 0.02$. Figs. 4–6 show the energy obtained by performing a spherical mean, for different values of ρ/λ_0. It is worthwhile to point out that these solutions do not correspond strictly to the thermal solution because the cases presented are quite distant from the case of validity of the thermal equation implying $\rho/\lambda_0 = \infty$, which, in absence of damping, would have a flat trend along the whole structure. Moreover, one can observe that, by increasing the ratio ρ/λ_0, the error decreases (Fig. 2) but the result tends to become flatter (Figs. 4–6), somehow loosening the expected information on the whole energy field.

Let us now establish some important differences related to the direct and reverberant fields. The direct field is the one generated by a point source with $A(\theta) = A_0$ (constant). By substituting this value in $\langle \hat{T}_{int} \rangle_\partial$ and $\langle \hat{T}_{coin} \rangle_\partial$, one can observe that none of these terms is equal to zero. On the contrary, referring to a diffuse field, one must think in statistical terms and estimate the expected

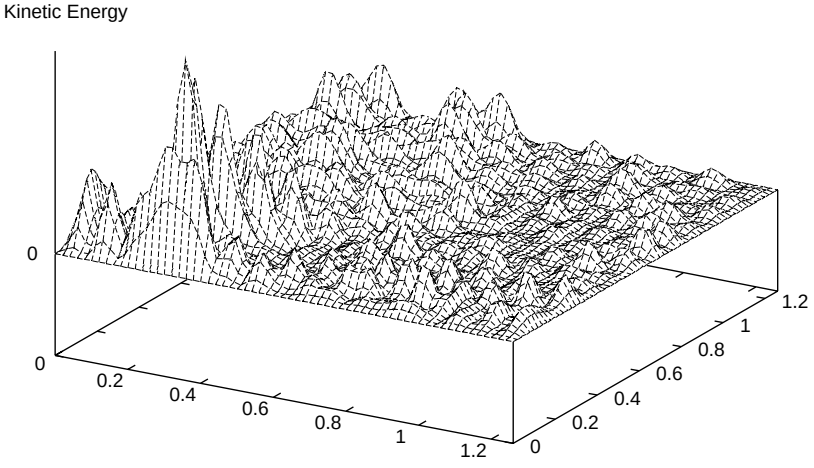

Fig. 3. Physical kinetic energy of the membrane.

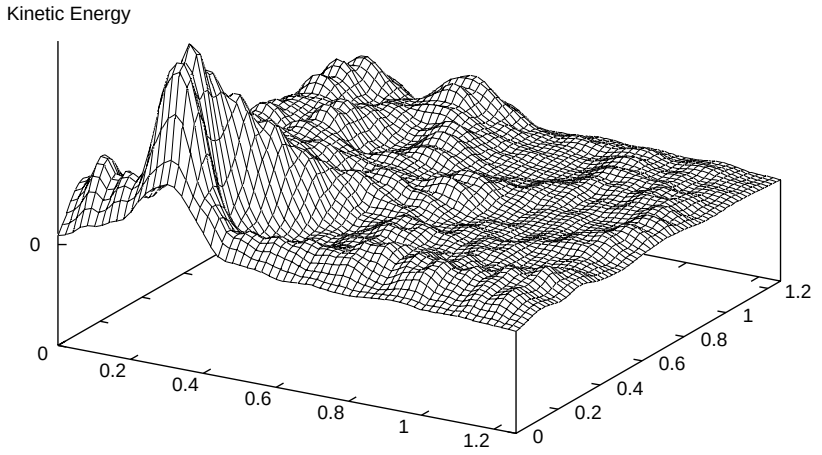

Fig. 4. Spherical mean of the kinetic energy: $\rho/\lambda_0 = 0.5$.

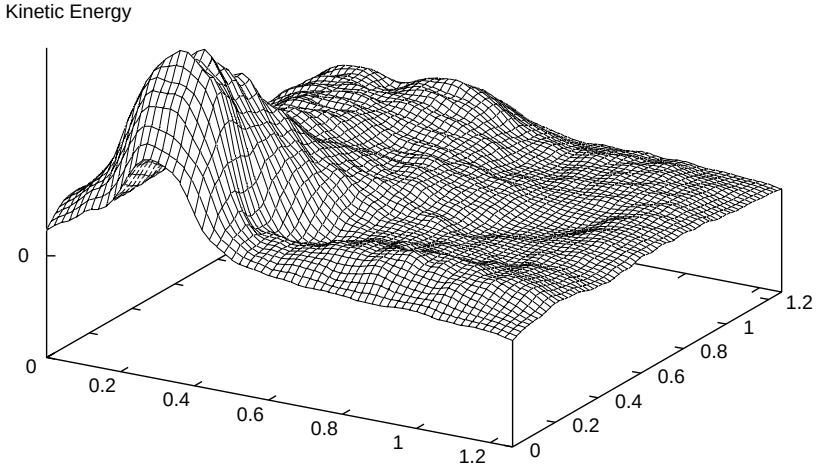

Fig. 5. Spherical mean of the kinetic energy: $\rho/\lambda_0 = 1$.

value $E[\langle \hat{T} \rangle_\partial]$ of the spherical mean. For a reverberant field one can assume that [3]

$$E\{A(\theta_1)A^*(\theta_2)\} = |A(\theta_1)|^2\delta(\theta_1 - \theta_2)$$

so that, by applying the average operator E to $\langle \hat{T} \rangle_\partial$, one can write

$$\begin{aligned} E[\langle \hat{T} \rangle_\partial] &= E[\langle \hat{T}_{int} \rangle_\partial + \langle \hat{T}_{coin} \rangle_\partial] \\ &= E[\langle \hat{T}_{int} \rangle_\partial] + E[\langle \hat{T}_{coin} \rangle_\partial]. \end{aligned}$$

Since $E[\langle \hat{T}_{int} \rangle_\partial] = 0$ because $\theta_1 \neq \theta_2$, one finally has

$$E[\langle \hat{T} \rangle_\partial] = E[\langle \hat{T}_{coin} \rangle_\partial]$$

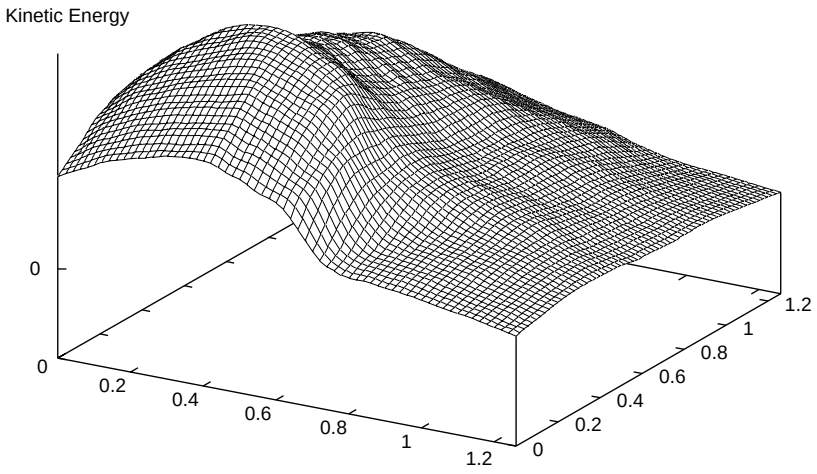

Fig. 6. Spherical mean of the kinetic energy: $\rho/\lambda_0 = 3$.

so that

$$\nabla^2\{E[\langle \hat{T}_{coin} \rangle_\partial]\} = 0.$$

In conclusion one can stress that for a general field, the interference terms in the energy balance equation are given by Φ_{int}, which tends to zero only asymptotically when performing spherical means. However, in the particular case of reverberant fields, $E[\langle \hat{T}_{int} \rangle_\partial] = 0$, so that the expected value of the spherical mean of the interference term is necessarily equal to zero.

4. Validity conditions for the heat modified equation

Two different two-dimensional systems will be analyzed: the first one concerns the case of the membrane, while the second one refers to a flexural plate.

It has already been mentioned that if the displacement field is represented as a superposition of equal plane waves ($A(\theta) = A$), then $w(\mathbf{r}) = 2\pi A J_0(k_0 r)$. The associated time-average kinetic energy density can be expressed as

$$\begin{aligned}\langle T \rangle &= \kappa_T w^2 = \kappa_T A^2 \int_0^{2\pi} \int_0^{2\pi} e^{jk_0\mathbf{n}(\theta_1)\mathbf{r}} e^{-jk_0\mathbf{n}(\theta_2)\mathbf{r}}\, d\theta_1\, d\theta_2 \\ &= \kappa_T 4\pi^2 A^2 J_0^2(k_0 r).\end{aligned}$$

It can be shown that such energy verifies Eq. (9).

For the membrane, it was shown in Section 2.1.1. that the time-average potential energy density is given by

$$\langle U \rangle = \frac{|grad\langle T \rangle|^2}{4k_0^2 \langle T \rangle}.$$

By developing the gradient, one obtains

$$\langle U \rangle = \frac{4\pi^2 A^2 \kappa_T \mathrm{J}_0'^2}{k_0^2}.$$

The total time-average energy density is then

$$\langle E \rangle = \langle T \rangle + \langle U \rangle = 4\pi^2 A^2 \kappa_T \left[\frac{\mathrm{J}_0'^2}{k_0^2} + \mathrm{J}_0^2\right].$$

It can be easily proved that such quantity does not satisfy the modified heat equation (14). However, if it is assumed that $r \gg \lambda_0$, i.e., $k_0 r \gg 1$, the Bessel function J_0 can be simplified by using an asymptotic expansion to yield

$$\mathrm{J}_0(k_0 r) \simeq \sqrt{\frac{2}{\pi k_0 r}} \cos\left(k_0 r - \frac{\pi}{4}\right)$$

so that

$$\mathrm{J}_0'(k_0 r) \simeq -\frac{1}{2}\sqrt{\frac{2}{\pi k_0 r^3}} \cos\left(k_0 r - \frac{\pi}{4}\right) - \sqrt{\frac{2}{\pi k_0 r}} k_0 \sin\left(k_0 r - \frac{\pi}{4}\right).$$

Because $k_0 r \gg 1$ one can neglect the first term on the r.h.s. with respect to the second one, so that one finally has

$$\langle E \rangle = \frac{8\pi A^2 \kappa_T}{k_0} \frac{1}{r}$$

i.e., the average total energy density is inversely proportional to r, in the hypothesis of large r.

It can now be immediately verified that this simplified expression satisfies the modified equation (14). Therefore, one can finally conclude that, for the membrane, the modified heat equation holds under the following conditions:

- time-average;
- large values of r with respect to λ_0;
- the plane wave interference is not neglected.

The case of the flexural plate is now considered. By assuming again that the displacement field is still represented by a superposition of plane waves, the expression of the potential energy density is completely different and given, in cylindrical co-ordinates, by

$$\langle U \rangle = \frac{D}{4}\left\{\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right)^2 - 2(1-\mu)\frac{\partial^2 w}{\partial r^2}\frac{1}{r}\frac{\partial w}{\partial r}\right\},$$

where $D = Eh^3/12(1-\mu^2)$, E and μ being Young's and the Poisson modulus, respectively, and h the plate thickness.

Using as before the asymptotic expansion of the Bessel function and keeping only the dominant terms in the expression of $\langle U \rangle$, one determines

$$\langle E \rangle = \langle T \rangle + \langle U \rangle = \frac{8\pi A^2 \kappa_T}{k_0} \frac{1}{r}\left(1 + \frac{Dk_0^4}{4\kappa_T}\right)\cos^2\left(k_0 r - \frac{\pi}{4}\right)$$
$$= \frac{16\pi A^2}{r}\frac{\kappa_T}{k_0}\cos^2\left(k_0 r - \frac{\pi}{4}\right).$$

It is easily proved that this expression does not satisfy Eq. (14). Therefore, for its validity, Eq. (14) requires another operation on the time-average energy density. One can observe that the trend of the energy density behaves like $1/r$ so that a simple moving space-average (e.g., a spherical mean) can be performed to eliminate the oscillating component, thus satisfying the modified heat conductivity equation. Thus, beyond the conditions stated above, the validity of the modified heat equation is reached, for flexural plates, under the further requirement of

- performing a moving space-average.

It can be stressed that, in the case of propagating waves instead of steady vibrations, the validity of Eq. (14) is obtained under less restricted requirements for both membranes and plates.

In this case a different superposition technique [17,18] leads to the solution:

$$w(r,t) = 2\pi A \mathrm{Re}\{H_0^{(2)}(k_0 r)\mathrm{e}^{\mathrm{j}\omega t}\},$$

where $\mathrm{H}_0^{(2)}$ is the zeroth order Hankel function of the second kind. Its asymptotic value is given by

$$\mathrm{H}_0^{(2)}(k_0 r) \simeq \sqrt{\frac{2}{\pi k_0 r}}\mathrm{e}^{-\mathrm{j}(k_0 r - \pi/4)}.$$

When considering the instantaneous value of the kinetic and potential energies, one obtains simply

$$T = \frac{4\pi A^2 \rho h \omega^2}{k_0 r}\sin^2\left(\omega t - k_0 r + \frac{\pi}{4}\right),$$

$$U = \frac{4\pi A^2 D k_0^4}{k_0 r}\cos^2\left(\omega t - k_0 r + \frac{\pi}{4}\right).$$

Therefore,

$$E = T + U = \frac{16\pi A^2 \kappa_T}{k_0 r}.$$

It satisfies Eq. (14), so that the modified heat equation is valid for propagating waves without performing any time- or space-averages.

5. Physical remarks on the interference contribution and conclusions

Based on the results presented so far and those exposed in Refs. [3,15,16], the following conclusions can be drawn:

(i) For a general three-dimensional system, governed by the Navier equation, it is not possible to determine an exact equation in terms of the energy density only. A general equation can be determined as a function of $\langle T \rangle$ and the displacement field w, whose last dependency cannot be eliminated [15].

(ii) For a two-dimensional structure governed by the Helmholtz equation, an energy equation (7) can be obtained, and the interference terms identified.

(iii) In Section 3.2 it has been shown that, as $\rho \to \infty$, $\Phi_{int} \to 0$: therefore from Eq. (17), one has that $\nabla^2 \langle \hat{T} \rangle = 0$, i.e., the energy equation tends to become thermal in absence of damping. Therefore, the vibration conductivity equation can be obtained as a limit space-average process, i.e., when the ratio $\rho/\lambda_0 \to \infty$ and damping is negligible. Accordingly with the experimental results given in Ref. [16] in presence of damping, for two-dimensional systems, even if $\rho/\lambda_0 \gg 1$, only the coincidence energy becomes thermal, while the interference energy is not. In fact, when damping is present, the wave field tends to be more direct and less reverberant so that the interference terms would not disappear completely, as also shown in Section 3.2. Thus, the space average does not eliminate in general the interference terms, but only smooths out their contribution. However, when considering the special case of a reverberant field (implying low damping), the expected value of the interference energy is zero, thus allowing one to accept the thermal equation for the energy statistical mean without performing the limit space average. This point is also somehow validated by considerations developed by Langley in Ref. [3] when discussing the validity of the thermal analogy. He observes that, by performing a local average on the displacement field expressed as a superposition of plane waves, the interference terms are expected to go to zero either at high frequencies or when a reverberant field is considered. However, if a direct field is analyzed, the space average is not sufficient to make these terms equal to zero.

(iv) The modified equation provided by Le Bot, Ichchou and Jezequel for cylindrical symmetry holds under particular conditions: for the membrane, it does not need space average, but time average and the condition of being at a far distance from the source ($r \gg \lambda_0$). For the plate, the further requirement of performing a moving space-average is needed. One can add that, when considering cylindrical propagating waves (instead of steady vibrations), the heat-modified equation does imply neither time nor space averages and it is verified by the instantaneous values of the energy field. Finally, it is worthwhile to point out that in the heat modified equation the average effect of plane wave interference is maintained. In fact the Bessel function, used in Section 4 to demonstrate the validity of the modified heat equation, can be considered as a superposition of interfering plane waves, as seen in Section 3.1.

The integral approach used by Le Bot (not restricted to the case of cylindrical symmetry) leads to more general applications but needs, for its validity, another assumption:

- the interference of plane waves derived from the primary and secondary sources (reflected waves) from the structure boundaries must be neglected.

This statement is possible when, at high frequencies [13],

- the excitation forces are uncorrelated;
- the modal behaviour of the system is not relevant: this condition holds especially when the modal overlap is high.

However, these last two conditions are those under which the thermal approach can be determined. Therefore, it could be concluded that there are no relevant differences between the validity conditions of these two methods, although in practice their developments are quite different. Maybe, an important difference can be stressed. In fact a high modal overlap implies any of the following conditions:

- the modes are rather close;
- the damping is rather high;
- high frequencies.

On the contrary vibration conductivity does not imply high damping, because the reverberant condition is more easily verified, thus eliminating the effect of the direct field when damping is low. Therefore, the thermal analogy would hold at high frequencies when the damping is low, while the modified heat equation would be valid, at high frequencies, for high damping. In any case, a high modal overlap and high frequencies are conditions under which one can be confident on SEA: it would be apparent that the thermal analogy and the modified heat equation are reliable under the conditions of the SEA validity.

A final comment concerns the comparisons of the three energy equations investigated in the case of cylindrical symmetry: the exact equation (9), the thermal equation (13) and the modified vibration conductivity equation (14). They express an energy balance referred to different space regions. This difference can be interpreted in terms of the ratio ρ/λ_0. Eq. (9) is an exact local (point) balance thus implying $\rho/\lambda_0 \to 0$. On the other hand the thermal equation is an energy balance obtained by averaging on a very large region, as implied by the condition $\rho/\lambda_0 \to \infty$. The modified heat equation can be regarded as an intermediate case in which the average is performed over a finite region. These statements have a physical counterpart in terms of waves' interference. In the exact equation the interference energy terms are clearly present. In the heat-modified equation, being the average performed over a finite region, the interference terms are still present but become more simple, as apparent by comparing Eqs. (9) and (14). Finally, when the region over which the average is performed tends to infinity the interference terms definitely disappear and one has the thermal equation.

While the forms of both the exact and thermal equations can be determined independent of any special assumption of cylindrical symmetry (see Eqs. (7) and (11)), the heat-modified equation is obtained only in this particular case. However, the integral formulation given in by Le Bot in Ref. [14] is a generalization of the heat-modified equation, valid for any considered wave pattern, although in this case the interference of primary and secondary sources must be neglected to obtain his integral equations.

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